Period 1, March 21, 2025

2012 (part c)

Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

$F(0) = \lim_{x \to 0} F(x)$

(a) Show that f is continuous at $x = 0$.

$\lim_{x\to 0} F(x)$ MUST EXIST, $F(0)$ MUST EXIST

(b) For $x \ne 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

(c) Find the average value of f on the interval $[-1, 1]$.

ⓐ $\lim_{x\to 0^-}(1-2\sin x) = 1-2\sin 0 = 1-2\cdot 0 = 1$

$\lim_{x\to 0^+}(e^{-4x}) = e^{-4\cdot 0} = e^0 = 1$

$\lim_{x\to 0} F(x) = 1$

$F(0) = 1-2\sin 0 = 1$

$F(0) = \lim_{x\to 0} F(x)$

ⓑ $F'(x) = \begin{cases} 0 - 2\cos x & \text{For } x \le 0 \\ e^{-4x}\cdot -4 & \text{For } x > 0 \end{cases}$

$-2\cos x = -3 \Rightarrow \cos x = \frac{3}{2}$ NOT Possible

$\frac{-4e^{-4x}}{-4} = \frac{-3}{-4} \Rightarrow e^{-4x} = \frac{3}{4} \Rightarrow \ln e^{-4x} = \ln\frac{3}{4}$

$-4x \cancel{\ln e} = \ln\frac{3}{4}$

$\frac{\cancel{-4}x}{\cancel{-4}} = \frac{\ln 3 - \ln 4}{-4}$

ⓒ $\frac{1}{1-(-1)}\int_{-1}^{1} F(x)\,dx \Rightarrow \frac{1}{2}\left[\int_{-1}^{0}(1-2\sin x)\,dx + \int_0^1 e^{-4x}\,dx\right]$

$\int e^{-4x}\,dx$

$u = -4x$

$du = -4\,dx$

$\frac{du}{-4} = dx$

$\int e^u \cdot \frac{du}{-4} = -\frac{1}{4}\int e^u\,du$

$-\frac{1}{4}e^u + C = -\frac{1}{4}e^{-4x}$

$\frac{1}{2}\left[(x + 2(\cos x))\Big|_{-1}^{0} + -\frac{1}{4}e^{-4x}\Big|_0^1\right]$

$0 + 2\cos 0 - [-1 + 2\cos(-1)]$

$0 + 2\cdot 1$

$2 + 1 - 2\cos(-1)$

$3 - 2\cos 1$

$-\frac{1}{4}e^{-4(1)} - \left(-\frac{1}{4}e^{-4\cdot 0}\right)$

$-\frac{1}{4e^4} + \frac{1}{4}e^0$

$-\frac{1}{4e^4} + \frac{1}{4}$

$\frac{1}{2}\left[3 - 2\cos(1) - \frac{1}{4e^4} + \frac{1}{4}\right]$

12. At $x=4$, the function given by $h(x)=\begin{cases} x^2, & x\le 4 \\ 4x, & x>4 \end{cases}$ is

(A) ~~Undefined~~
(B) Continuous but not differentiable
(C) ~~Differentiable but not continuous~~
(D) Neither continuous nor differentiable
(E) Both continuous and differentiable

$\lim_{x\to 4} x^2 = 16$

$\lim_{x\to 4} 4x = 16$

$F(4)=16$

Continious

$$h'(x) = \begin{cases} 2x & x\le 4 \\ 4 & x>4 \end{cases}$$

$\lim_{x\to 4} 2x = 8$
$\lim_{x\to 4} 4 = 4$
→ Not The Same
"Not differentiable"

If f is a differentiable function, then $f'(a)$ is given by which of the following?

I. $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ = ✓

II. $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ = ✓

III. $\lim_{x\to a} \dfrac{f(\cancel{x+h})-f(x)}{h}$ → not going to zero

$\lim_{h\to 0} \dfrac{F(x+h)-F(x)}{h} = F'(x)$

$\lim_{x\to a} \dfrac{F(x)-F(a)}{x-a} = F'(a)$

(A) I only (B) II only (C) I and II only (D) I and III only (E) I, II, and III

7. If $a \neq 0$, then $\lim\limits_{x \to a} \dfrac{x^2 - a^2}{x^4 - a^4}$ is $= \dfrac{1}{x^2+a^2}$

(A) $\dfrac{1}{a^2}$ (B) $\dfrac{1}{2a^2}$ (C) $\dfrac{1}{6a^2}$ (D) 0 (E) nonexistent

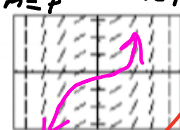
$\lim\limits_{x \to a} \dfrac{2\cancel{x}}{4x^{\cancel{3}2}} = \dfrac{2}{4a^2} = \dfrac{1}{2a^2}$

$\lim\limits_{x \to a} \dfrac{x^2 - a^2}{x^4 - a^4} = \lim\limits_{x \to a} \dfrac{\cancel{x^2 - a^2}}{(\cancel{x^2-a^2})(x^2+a^2)} = \dfrac{1}{a^2+a^2}$

$\dfrac{1}{2a^2}$

8.

x=− y=+ m=+ x=+ y=+ m=+



m=+ x=− y=− Fits

The slope field for a certain differential equation is shown above. Which of the following could be a specific solution to that differential equation?

(A) $y = \sin x$ (B) $y = \cos x$ (C) $y = x^2$ (D) $y = \frac{1}{6}x^3$ (E) $y = \ln x$

~~$\frac{dy}{dx} = \cos x$~~ ~~$\frac{dy}{dx} = -\sin x$~~ ~~$\frac{dy}{dx} = 2x$~~ $\frac{dy}{dx} = \frac{1}{2}x^2$ ~~$\frac{dy}{dx} = \frac{1}{x}$~~

$\cos\frac{2\pi}{3} = -\frac{1}{2}$ $-\sin\frac{\pi}{6} = -\frac{1}{2}$ $2(-1) = -2$ always be + $\frac{1}{-1} = -1$

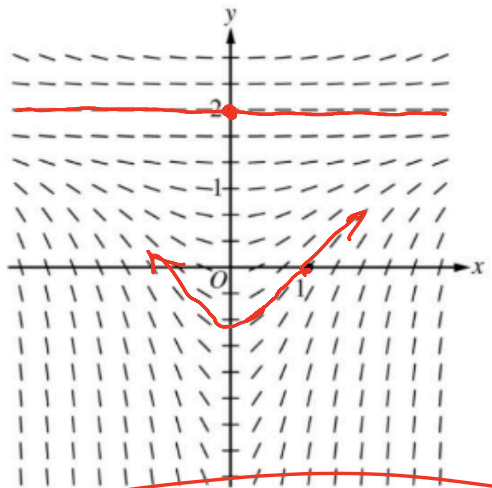
6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point (0, 2), and sketch the solution curve that passes through the point (1, 0).

$(0,-2)$

$\frac{dy}{dx} = \frac{1}{3}(0)(-2-2)^2$

$= 0$



$(1,0)$

(b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

b. Slope and Point

$\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$. $(1,0)$

$\frac{1}{3}(1)(0-2)^2$

$\frac{1}{3} \cdot 4 = \frac{4}{3} = m$

$y - 0 = \frac{4}{3}(x-1)$

$y = \frac{4}{3}(x-1)$

$\frac{4}{3}(0.7-1) = \frac{4}{3} \cdot (-0.3) = \frac{4}{\cancel{3}} \cdot \frac{-\cancel{3}}{10}$

$\frac{-4}{10} = \frac{-2}{5} = -0.4$

$\boxed{\frac{-1}{y-2} = \frac{1}{6}x^2 + \frac{2}{6}}$

$\frac{\cancel{dx}}{(y-2)^2 \cancel{dx}} \frac{dy}{} = \frac{1}{3}x\cancel{(y-2)^2} \cdot \frac{dx}{\cancel{(y-2)^2}} \Rightarrow \int \frac{1}{(y-2)^2}dy = \int \frac{1}{3}x\,dx \Rightarrow -(y-2)^{-1} = \frac{1}{6}x^2 + C$

$u = y-2$

$du = dy$

$\int u^{-2}du$

$\frac{1}{-1}u^{-2+1=-1}$

$\frac{-1}{y-2} = \frac{1}{6}x^2 + C \leftarrow (1,0)$

$\frac{-1}{0-2} = \frac{1}{6}(1)^2 + C \Rightarrow \frac{1}{2} = \frac{1}{6} + C = \frac{2}{6}$

$$(y-2)\cdot\frac{-1}{y-2}=\left(\frac{1}{6}x^2+\frac{2}{6}\right)(y-2)$$

$$\frac{6\cdot -1}{x^2+2}=\frac{\cancel{\frac{1}{6}(x^2+2)}(y-2)\cancel{\cdot 6}}{\cancel{x^2+2}}$$

$$\frac{-6}{x^2+2}=y-2$$
$$+2 \qquad +2$$

$$\frac{-6}{x^2+2}+2=y$$

9. If $\frac{dy}{dt}=-2y$ and if $y=-1$ when $t=0$, what is the value of t for which $y=-\frac{1}{2}$?

(A) $-\frac{\ln 2}{2}$ (B) $-\frac{1}{4}$ (C) $\frac{\ln 2}{2}$

(D) $\frac{\sqrt{2}}{2}$ (E) $\ln 2$

$$\frac{\cancel{dT}\cdot}{y}\frac{dy}{\cancel{dT}}=\frac{-2\cancel{y}\cdot dT}{\cancel{y}}$$

$$\int\frac{1}{y}dy=\int -2\,dT$$

$\ln|y|=-2T+C$

$\ln|-1|=-2(0)+C$

$\ln 1=0+C$

$0=0+C\Rightarrow C=0$

$\ln|y|=-2T\Rightarrow e^{-2T}=y$

$\ln\frac{1}{2}$ $\qquad e^{-2T}=-\frac{1}{2}$

$\ln 1-\ln 2=-2T$ $\qquad \ln e^{-2T}=\cancel{\ln -\frac{1}{2}}$

$$\frac{0+\ln 2}{-2}=\frac{-2T}{-2}$$

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal

$H=91$

temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato

← Temp of Room

is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

$T=0 \quad H=91$

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

Point $(0,91)$ $\quad -\frac{1}{4}(91-27) = \frac{-64}{4} = -16 = m$

Find Line $\quad$ Plug in $T=3$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$$\frac{dH}{dT} = -\frac{1}{4}H + \frac{27}{4}$$

$H > 27$

$$\frac{d^2H}{dT^2} = -\frac{1}{4}\frac{dH}{dT} + 0 \Rightarrow -\frac{1}{4}\left(-\frac{1}{4}H + \frac{27}{4}\right) = -\frac{1}{4}\cdot - = +$$

concave up

under estimite

$$\frac{\cancel{dT}\,dG}{(G-27)^{2/3}\,\cancel{dT}} = \frac{-\cancel{(G-27)^{2/3}}}{\cancel{(G-27)^{2/3}}} \Rightarrow \int (G-27)^{-2/3}\,dG = \int -1\,dT \Rightarrow 3(G-27)^{\frac{1}{3}} = -T + C$$

$(0,91)$

$$\int (G-27)^{-2/3}\,dG = \int U^{-2/3}\,dU = 3U^{\frac{1}{3}} = 3(G-27)^{\frac{1}{3}}$$

$U = G-27$

$dU = dG$

$\frac{3}{1}\cdot U \quad -2/3 + 1 = \frac{1}{3}$

$3(91-27)^{\frac{1}{3}} = -0 + C$

$3\cdot(64^{\frac{1}{3}})$

$3\cdot 4 = C = 12$

$$\frac{3(G-27)^{1/3}}{3} = \frac{-T+12}{3}$$

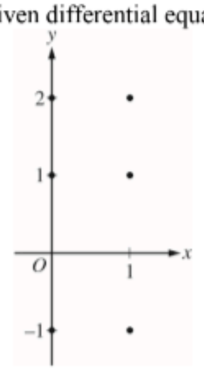
$$\left(\left(G-27\right)^{1/3}\right)^{3}=\left(\frac{-T+12}{3}\right)^{3}$$

$$\begin{aligned} G-\cancel{27} &= \left(\frac{-T+12}{3}\right)^{3}+27 \\ +\cancel{27} & \end{aligned}$$

$$G=\left(\frac{-T+12}{3}\right)^{3}+27$$

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

$y = 2x - 2$

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$\frac{dy}{dx} = 2x - y = m$

$2x - (mx + b) = m$

$2x - mx - b = m$

$2x - mx - b - m = 0$

$x(2 - m) - 1(m + b) = 0$ ← No x's

$2 - m = 0$

$m = 2$

$-1(2 + b) = 0$

$b = -2$

1. If $x^2 + y^2 = 25$, what is the value of $\frac{d^2y}{dx^2}$ at the point $(4,3)$?

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y} = \frac{-x}{y}$$

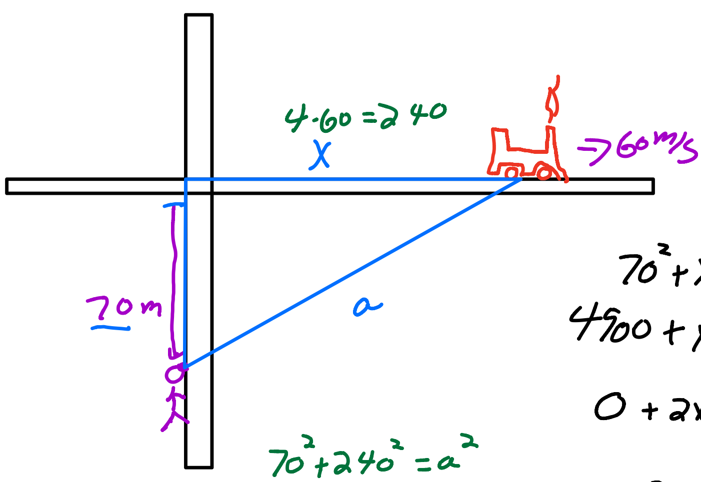
$$\frac{d^2y}{dx^2} = \frac{-1 \cdot y - (-x)\cdot\frac{dy}{dx}}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = \text{clean up}$$

$$\frac{\frac{-y}{1} - \frac{x^2}{y}}{y^2}$$

3. A railroad track and a road cross at right angles. An observer stands on the road 70 meters south of the crossing and watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

$\frac{dA}{dT}$ $\frac{d}{dT}$

(A) 57.60 (B) 57.88 (C) 59.20 (D) 60.00 (E) 67.40



$$\frac{dx}{dT} = 60\ \text{m/s}$$

$$70^2 + 240^2 = a^2$$

$$250 = a$$

$$70^2 + x^2 = a^2$$

$$4900 + x^2 = a^2$$

$$0 + 2x\frac{dx}{dT} = 2a\frac{dA}{dT}$$

$$2 \cdot 240\text{m} \cdot 60\,\text{m/s} = 2(250\text{m}) \cdot \frac{dA}{dT}$$